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LETTER TO THE EDITOR

An exact result for the Kagomé lattice Ising model with magnetic field

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Abstract. An anisotropic Ising model with three interaction parameters K_1, K_2, K_3 and a magnetic field H, formulated on the Kagomé lattice, is solved exactly for an appropriate relation between K_i and H. When this relation is satisfied the system becomes equivalent to a free-fermion model.

There are very few exact results for the two-dimensional Ising model with a magnetic field. When the magnetic field $H = \mathrm{i}\pi/2$, exact solutions have been obtained for the square lattice (Yang and Lee 1952, McCoy and Wu 1967), for the triangular and honeycomb lattices (Baxter 1965) and for a more general Ising model with multispin interactions (Wu 1986). Other exact solutions have been obtained by Fisher (1960) for a decorated square lattice where the magnetic field interacts only with the decorating spins (super-exchange model).

More recently, a new type of exact result for Ising-type models with magnetic field has been found, the so-called disorder solutions (Stephenson 1970, Verhagen 1976, Enting 1977a, b, Jaekel and Maillard 1985, Wu 1985, Rujan 1987). For this class of solutions the model trivialises because a reduction of dimensionality occurs in such a way that it becomes equivalent to a zero- or one-dimensional system. Also, the exact solution of the hard-hexagon model (Baxter 1980) represents an important result for the Ising model. This lattice gas system is a particular case of the triangular lattice Ising model with a magnetic field.

In this letter an exact solution for an anisotropic Ising model with a magnetic field, formulated on the Kagomé lattice (KL), is outlined. The Boltzmann weight of the model, associated with the elementary cell of the KL, is as follows:

$$\begin{aligned} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) &= \exp[K_1\sigma_5(\sigma_2+\sigma_4) + K_2\sigma_5(\sigma_1+\sigma_3) + K_3(\sigma_1\sigma_2+\sigma_3\sigma_4) \\ &\quad + H\sigma_5 + H/2(\sigma_1+\sigma_2+\sigma_3+\sigma_4)] \end{aligned} \tag{1}$$

where the Ising spins σ_i and the interaction parameters K_i are depicted in figure 1.

The partition function of the model is given by

$$Z_{\mathrm{Kag}}(K_1, K_2, K_3, H) = \sum_{(\sigma)} \Pi W \tag{2}$$

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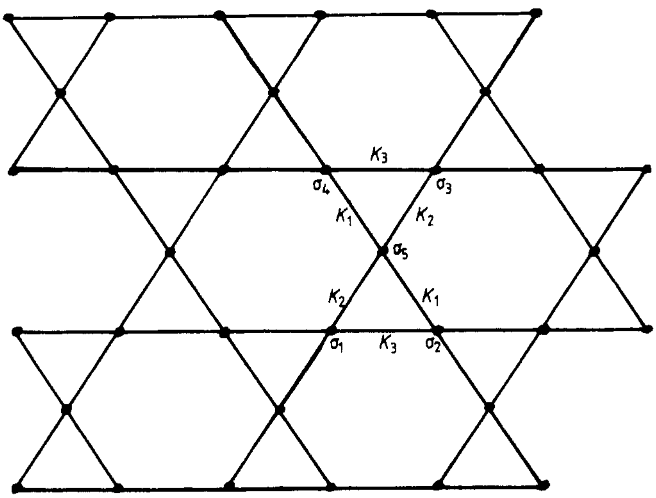

Figure 1. The Kagomé lattice shown in an elementary cell with the Ising spins σ_i associated with each site and the three interaction parameters K_1, K_2, K_3.

where the sum is performed over all spin configurations and the products are taken over all elementary cells of a lattice with $3N$ sites and periodic boundary conditions.

The procedure employed to obtain the exact result mentioned above is very simple. It is based on the anisotropic generalisation of the decoration-decimation transformation which is used to map the isotropic KL Ising model with a magnetic field on the isotropic honeycomb lattice Ising model with a magnetic field (Syozi 1972).

As the first step, the KL is decorated by introducing an Ising spin S_i on the centre of each elementary triangle of the lattice, as indicated in figure 2. In this way the Boltzmann weight (1) can be expressed as follows:

$$\begin{aligned} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) &= R^2 \sum_{S_1, S_2} \exp[S_1(M_1\sigma_3 + M_2\sigma_4 + M_3\sigma_5) + S_2(M_1\sigma_1 + M_2\sigma_2 + M_3\sigma_3) \\ &\quad + H\sigma_5 + H/2(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)]. \end{aligned} \tag{3}$$

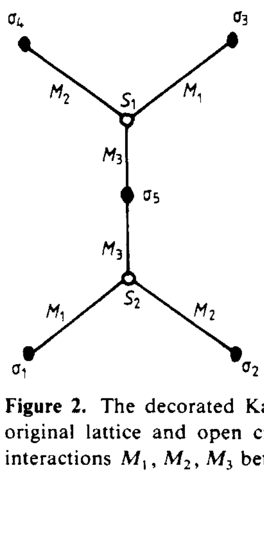

Figure 2. The decorated Kagomé lattice. Full circles indicate the Ising spins σ_i of the original lattice and open circles represent the decorating Ising spins S_1 and S_2. The interactions M_1, M_2, M_3 between both types of spins are also shown.

The parameters M_i and R are related to the K_i by the well known expressions of the star-triangle relation (Baxter 1982)

$$\sinh(2K_i)\sinh(2M_i) = \alpha^{-1} \qquad i = 1, 2, 3 \tag{4}$$

where

$$\alpha = (1+t_1^2)(1-t_2^2)(1-t_3^2)[16(1+t_1t_2t_3)(t_1+t_2t_3)(t_2+t_1t_3)(t_3+t_1t_2)]^{-1/2} \tag{5}$$

with

$$t_i = \tanh(K_i) \qquad i = 1, 2, 3$$

$$\cosh(2M_i) = \cosh(2K_j)\cosh(2K_l) + \sinh(2K_j)\coth(2K_i) \tag{6}$$

and permutations of (6) with respect to i, j, l

$$R^2 = \alpha^2/2\sinh(2K_1)\sinh(2K_2)\sinh(2K_3). \tag{7}$$

The partition function of the KL model can now be expressed in terms of the partition function of the decorated model as follows:

$$Z_{\mathrm{Kag}}(K_1, K_2, K_3, H) = R^{2N} Z_{\mathrm{dec}}(M_1, M_2, M_3, H). \tag{8}$$

In this way the spins σ_i become decoupled and can be summed up. After the decimation of the spins σ_i, the resulting model is the Ising model on the honeycomb lattice with $2N$ sites, interaction parameters L_1, L_2, L_3 and a magnetic field $\bar{H}$. Therefore the partition function of the KL model is expressed finally in terms of the partition function of the honeycomb lattice model:

$$Z_{\mathrm{Kag}}(K_1, K_2, K_3, H) = (R^2 A_1 A_2 A_3)^N Z_{\mathrm{honey}}(L_1, L_2, L_3, \bar{H}) \tag{9}$$

where $L_1, L_2, L_3, A_1, A_2, A_3$ and $\bar{H}$ are given by

$$\exp(4L_i) = \cosh(2M_i + H)\cosh(2M_i - H)[\cosh(H)]^{-2} \qquad i = 1, 2, 3 \tag{10}$$

$$\bar{H} = H_1 + H_2 + H_3 \tag{11}$$

with

$$\exp(4H_i) = \cosh(2M_i + H)[\cosh(2M_i - H)]^{-1} \qquad i = 1, 2, 3 \tag{12}$$

$$A_i = 2[\cosh(2M_i + H)\cosh(2M_i - H)\cosh^2(H)]^{1/4} \qquad i = 1, 2, 3. \tag{13}$$

When $K_1 = K_2 = K_3$ (isotropic case) one reobtains the known expressions that relate the isotropic KL model with the isotropic honeycomb lattice model, both with magnetic field (Syozi 1972).

Let us recall that the Ising model on the honeycomb lattice is exactly soluble when the magnetic field $\bar{H} = 0$ (Syozi 1972) and when $\bar{H} = \mathrm{i}\pi/2$ (Baxter 1965). In both cases one has

$$\exp(4\bar{H}) = 1. \tag{14}$$

Therefore, taking into account (9), the Ising model on the KL with a magnetic field H is exactly soluble when condition (14) is satisfied. Obviously this condition is identically satisfied when $H = 0$, and it is in this form that the partition function of the anisotropic KL lattice without magnetic field has been obtained by Kano and Naya (1953).

However, and this is the fundamental point of this work, equation (14), when expressed in terms of a function of K_1, K_2, K_3 and H, also has non-trivial and interesting solutions with $H \neq 0$. By using the relations (4), (5), (6), (11) and (12), and after some algebra, condition (14) can be expressed, for finite H ($H \neq 0$ and $H \neq \infty$) as

$$\tanh^2(H) = \alpha^2/4[S_1S_2S_3 + C_1C_2C_3 + S_1 + S_2 + S_3 - \exp 4(K_1 + K_2 + K_3)] \tag{15}$$

where

$$S_i = \sinh(4K_i) \qquad C_i = \cosh(4K_i) \qquad i = 1, 2, 3$$

and α is given by (5).

It is evident that this equation has the same symmetries as the partition function (2), i.e. it is symmetric with respect to K_1, K_2, K_3 and is invariant under the transformations $H \to -H$, $H \to \pm H \pm \mathrm{i}\pi$ and $K_i \to K_i \pm \mathrm{i}\pi/2$, where each parameter can be transformed independently.

For arbitrary values of K_1, K_2, K_3, equation (15) determines a unique value of H for which the model can be solved exactly.

Let us recall that the honeycomb lattice Ising model with $\bar{H} = 0$ or $\bar{H} = \mathrm{i}\pi/2$ is a free-fermion model. Very recently it has been shown by Baxter (1986) that the partition function of the free-fermion model can be evaluated for lattices of arbitrary size. Therefore, when condition (15) is satisfied, the partition function (2) of the KL model can be calculated for arbitrary N.

When one of the parameters K_i is taken equal to zero ($K_3 = 0$, for example), the spins connected by this interaction can be decimated away. After this, the Ising model on a square lattice is obtained. However, the resulting magnetic field on this lattice is identically null when equation (15) is satisfied. Therefore the result presented in this letter cannot be extended to the other regular two-dimensional lattices. It is strongly dependent of the particular structure of the KL.

It is evident from (15) that if K_1, K_2, K_3 are real, the magnetic field H can be real, purely imaginary or with an imaginary part equal to $\mathrm{i}\pi/2$. Hence there are solutions of (15) in the physical region of the parameter space of the model.

Complex values of H are also interesting in order to gain some insight into the mathematical structure of the partition function. Besides, by using lattice model transformations, there exists the possibility to map the non-physical region of the present model in the physical region of other types of system. An example of this class is furnished by the square lattice Ising model with a magnetic field equal to $\mathrm{i}\pi/2$. By a duality transformation this model can be mapped on the 'fully frustrated' Villain model. For $K_1, K_2, K_3 > 0$, H is always purely imaginary. On the other hand, the real values of K_i that give real values of H are arranged in such a way that all the elementary triangles of the KL are frustrated (for the concept of frustration see Toulouse (1977)).

It follows from this result that the system can have critical points when condition (15) is fulfilled, because the result of Yang and Lee (1952) about the absence of phase transitions for the Ising model with magnetic field does not apply when there are antiferromagnetic interactions.

The magnetic field $\bar{H}$ of the honeycomb lattice can take the values 0 or $\mathrm{i}\pi/2$ when (15) holds, depending on the given values of the parameters K_i. In the isotropic case $K_1 = K_2 = K_3 = K$ condition (15) reduces to

$$\tanh^2(H) = 3[\tanh^2(2K) - 2\tanh(2K)]^{-1} \tag{16}$$

and for arbitrary real values of K, H always take complex values. If $K>0$, H is purely imaginary and for $K<0$, H has an imaginary part equal to $i\pi/2$. Moreover, in this isotropic case, $\bar{H}$ only take the value $i\pi/2$ for all real values of K.

For the general anisotropic situation it is possible, in principle, by using the above results, to evaluate the zero-field susceptibility per spin of the KL model

$$\chi_0(K_1, K_2, K_3) = (3NkT)^{-1} \frac{\partial^2}{\partial H^2} \log(Z_{\mathrm{Kag}}(K_1, K_2, K_3, H))|_{H=0} \tag{17}$$

where k is the Boltzmann constant and T is the temperature, when the parameters K_i satisfy the equation obtained from (15) in the limit $H \to 0$, i.e.

$$\exp[4(K_1+K_2+K_3)] = C_1C_2C_3 + S_1S_2S_3 + S_1 + S_2 + S_3. \tag{18}$$

In the more symmetric case $K_1 = K_2$ this condition can be expressed by means of a very simple relation as follows:

$$\exp(4K_3) = 2(2C_1 + S_1)^{-1}. \tag{19}$$

For arbitrary real values of K_1, the values of K_3 obtained from (19) are also real. Therefore χ_0 can be exactly evaluated in a physical region of the parameter space of the model.

Moreover, for this case ($K_1 = K_2$) the magnetic field H obtained from (15) is real for arbitrary real values of K_1 if K_3 satisfies the inequality

$$4[3\exp(4K_1) + \exp(-4K_1)]^{-1} < \exp(4K_3) < 2[\exp(4K_1)+1]^{-1}. \tag{20}$$

A fortunate fact is that, if the zero-field susceptibility χ_0 is known for the KL, it can also be calculated, through very simple expressions, for the honeycomb and triangular lattices (Syozi 1972). Therefore the results of this letter enable us to calculate χ_0 for the anisotropic Kagomé, honeycomb and triangular lattices, when an appropriate relation between the interaction parameters of the models is satisfied in each case.

It is worth pointing out that the surface defined by (18) in the parameter space K_1, K_2, K_3 is different from the critical variety and the disorder variety of the model with $H = 0$.

It is obviously also of interest to analyse the critical behaviour of this model in the general anisotropic case, when condition (15) is satisfied. This aspect and an explicit evaluation of χ_0 under the restriction (18) is now under consideration.

To summarise, an exact solution of the anisotropic KL Ising model with a magnetic field has been presented. The solution is valid when condition (15) between the parameters of the model is satisfied.

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